Sample PROJECT: COMPUTATION OF LINEARIZED EULER EQUATIONS by Lax Wendroff

This project solves the linearized Euler equations for flow past a thin airfoil. Flow is assumed to be uniform ($\rho = 1, u = M_\infty, v = 0$) at inflow and is used as the reference state for the local linearization.
We also simplify the equations by assuming constant temperature, i.e. Pressure $= \rho$.

Equations: The Euler equations for flow about a slender body in two dimensions can be written as (before linearization)

$$\frac{\partial Q}{\partial t} + \frac{\partial E(Q)}{\partial x} + \frac{\partial F(Q)}{\partial y} = 0 \tag{1}$$

with

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, E = \begin{pmatrix} \rho u \\ \rho u^2 + \rho \\ \rho u v \end{pmatrix}, \quad F = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + \rho \end{pmatrix} \tag{2}$$

Note: Q, E, F are 3×1 vectors.
We rewrite in quasi-linear form, using e.g.

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial Q}\frac{\partial Q}{\partial x} = A\frac{\partial Q}{\partial x}$$

gives us

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = 0 \tag{3}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -u^2+1 & 2u & 0 \\ -uv & v & u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2+1 & 0 & 2v \end{pmatrix} \tag{4}$$

To simplify the application we freeze A and B at the reference state $\rho = 1, u = M_\infty, v = 0$ to give us

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = 0 \tag{5}$$

now with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -M_\infty^2+1 & 2M_\infty & 0 \\ 0 & 0 & M_\infty \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & M_\infty \\ 1 & 0 & 0 \end{pmatrix} \tag{6}$$

Now we are working with a small disturbance form of the Euler equations where ρ, u and v are the pertubation components from a uniform flow in the x direction. The Mach number is M_∞ and the equations can be used to study subsonic to supersonic small disturbance flow over slender bodies or past surfaces with small surface variations. The matrix A has **real** distinct eigenvalues $M_\infty, M_\infty + 1, M_\infty - 1$ and B the eigenvalues $0, 1, -1$, so that the system is hyperbolic in time.

Geometry: The grid is uniform in $-1 \leq x \leq 3.0$ and $0 \leq y \leq 2$, although you can choose any limits you want. At the lower surface, a biconvex thin airfoil is used with

$$\begin{aligned} y_{wall} &= \quad \tau x(1-x)/2 \quad 0 \leq x \leq 1 \\ y_{wall} &= \qquad 0 \quad x \leq 0, x \geq 1 \end{aligned} \tag{7}$$

where τ is the thickness.
Boundary conditions:

1. At inflow ($x = -1$), fix $\rho u = M_\infty, \rho v = 0$, and set $\frac{\partial \rho}{\partial x} = 0$.
2. At the top ($y = 2$) fix all the variables $\rho = 1, u = M_\infty, v = 0$.
3. At outflow, ($x = 3$), use $\frac{\partial Q}{\partial x} = 0$.
4. Assume that v is specified at the lower boundary ($y = 0$) in x using thin airfoil conditions, that is
$$v = M_\infty \frac{dy_{wall}}{dx} \quad \text{imposed at} \quad y = 0 \tag{8}$$

Project: Lax-Wendroff Numerical Marching Method.
Discretize the field using a uniform grid with $x_{j,k} = (j-1)\Delta x$ and $y_{j,k} = (k-1)\Delta y$. Expand $\vec{Q}$ in time (i.e. $\vec{Q}^{n+1}$ about $\vec{Q}^n$) as

$$\vec{Q}^{n+1}_{j,k} = \vec{Q}^n_{j,k} + \Delta t \partial_t \vec{Q}^n_{j,k} + \frac{(\Delta t)^2}{2} \partial_{tt} \vec{Q}^n_{j,k} + \cdots \tag{9}$$

Now from Eq. 5

$$\partial_t \vec{Q} = -A\partial_x \vec{Q} - B\partial_y \vec{Q} \tag{10}$$

and

$$\partial_{tt}\vec{Q} = -A\partial_x(\partial_t\vec{Q}) - B\partial_y(\partial_t\vec{Q}) = -A\partial_x(-A\partial_x\vec{Q} - B\partial_y\vec{Q}) - B\partial_y(-A\partial_x\vec{Q} - B\partial_y\vec{Q}) \tag{11}$$

Therefore Eq. 9 can be written as

$$\begin{aligned} \vec{Q}^{n+1}_{j,k} = \quad & \vec{Q}^n_{j,k} - \Delta t \left(A\partial_x \vec{Q}^n_{j,k} + B\partial_y \vec{Q}^n_{j,k} \right) + \\ & \tfrac{(\Delta t)^2}{2} \left(A^2 \partial_{xx} \vec{Q}^n_{j,k} + B^2 \partial_{yy} \vec{Q}^n_{j,k} \right) + \\ & \tfrac{(\Delta t)^2}{2} \left(AB\partial_{xy} \vec{Q}^n_{j,k} + BA\partial_{yx} \vec{Q}^n_{j,k} \right) + \cdots \end{aligned} \tag{12}$$

Approximating the x, y derivatives with second order accurate central differences, this process leads to a numerical scheme which is referred to as the explicit one-step Lax

Wendroff scheme:

$$
\begin{aligned}
\vec{Q}^{n+1}_{j,k} = \vec{Q}^{n}_{j,k} - \quad & \Delta t \left(A \frac{\vec{Q}^n_{j+1,k} - \vec{Q}^n_{j-1,k}}{2\Delta x} + B \frac{\vec{Q}^n_{j,k+1} - \vec{Q}^n_{j,k-1}}{2\Delta y} \right) + \\
& \frac{(\Delta t)^2}{2} \left(A^2 \frac{\vec{Q}^n_{j+1,k} - 2\vec{Q}^n_{j,k} + \vec{Q}^n_{j-1,k}}{(\Delta x)^2} + B^2 \frac{\vec{Q}^n_{j,k+1} - 2\vec{Q}^n_{j,k} + \vec{Q}^n_{j,k-1}}{(\Delta y)^2} \right) + \\
& \frac{(\Delta t)^2}{2} \left(AB \frac{\vec{Q}^n_{j+1,k+1} - \vec{Q}^n_{j+1,k-1} - \vec{Q}^n_{j-1,k+1} + \vec{Q}^n_{j-1,k-1}}{4\Delta x \Delta y} \right) + \\
& \frac{(\Delta t)^2}{2} \left(BA \frac{\vec{Q}^n_{j+1,k+1} - \vec{Q}^n_{j-1,k+1} - \vec{Q}^n_{j+1,k-1} + \vec{Q}^n_{j-1,k-1}}{4\Delta x \Delta y} \right)
\end{aligned} \tag{13}
$$

Equation 13 can be integrated from the uniform initial condition $Q^{(0)} = \begin{pmatrix} 1 \\ M_\infty \\ 0 \end{pmatrix}$ to a steady state.

A sample Matlab code is provided for you. It uses the Lax Wendroff scheme. It is stable for $CFL \leq 1.0$. Try $CFL = 1.0$ and $nx = 10$, (nx is used to define the grid in x and y). I put nx points on the airfoil, then $\Delta x = 1.0/(nx-1)$, let $\Delta y = \Delta x$ and compute the total number of points for the problem. Study this code as a starting point for your choosen project. It produces plots of $C_p = (\rho - 1)/(0.5 * M_\infty^2)$ at the wall, some density contours, and residual history, $||R||_2$.