MIDTERM AA214A – October 29,2008

1. Taylor Tables

(a)

$$\Delta x(\delta_x u)_j + \Delta x\alpha(\delta_x u)_{j-1} - (u_j + Au_{j-1} + Bu_{j-2}) = \Delta x\ er_t$$

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	
$\Delta x(\delta_x u)_j$		1				
$\alpha\Delta x(\delta_x u)_{j-1}$		α	$\frac{(\alpha)(-1)^1}{1!}$	$\frac{(\alpha)(-1)^2}{2!}$	$\frac{(\alpha)(-1)^3}{3!}$	
$-u_j$	-1					
$-Au_{j-1}$	$-A$	$\frac{(-A)(-1)^1}{1!}$	$\frac{(-A)(-1)^2}{2!}$	$\frac{(-A)(-1)^3}{3!}$	$\frac{(-A)(-1)^4}{4!}$	
$-Bu_{j-2}$	$-B$	$\frac{(-B)(-2)^1}{1!}$	$\frac{(-B)(-2)^2}{2!}$	$\frac{(-B)(-2)^3}{3!}$	$\frac{(-B)(-2)^4}{4!}$	
$\Delta x er_t$	0	0	0	?		

where the first 3 columns set to zero will produce at least a 2nd order method. This results in three equations for the three unknowns, α, A, B

$$\begin{aligned} A + B + 1 &= 0 \\ 1 + \alpha + A + 2B &= 0 \\ \alpha + \frac{A}{2} + 2B &= 0 \end{aligned}$$

which gives $\alpha = -1, A = -2, B = 1$. The scheme is

$$(\delta_x u)_j - (\delta_x u)_{j-1} = \frac{1}{\Delta x}\left(u_j - 2u_{j-1} + u_{j-2}\right)$$

(b)

$$\Delta x\ er_t = \left(\frac{\alpha}{2} + \frac{A}{6} + \frac{8B}{6}\right)\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$$

giving $er_t = \frac{1}{2}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_j$ a 2nd order scheme.

2. Modified wave number problems

   (a) We apply $u_j = e^{ikj\Delta x}$ to

$$(\delta_x u)_j = \frac{1}{4\Delta x}\left(-u_{j+2} + 4u_{j+1} - 4u_{j-1} + u_{j-2}\right)$$

   and get

$$\left(ik^* e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(-e^{2ik\Delta x} + 4e^{ik\Delta x} - 4e^{-ik\Delta x} + e^{-ik\Delta x}\right)/(4\Delta x)$$

   which give us

$$ik^* = i\frac{(4\sin(k\Delta x) - \sin(2k\Delta x))}{2\Delta x}$$

   (b) For the compact differencing approximation of problem 1

$$(\delta_x u)_j - (\delta_x u)_{j-1} = \frac{1}{\Delta x}\left(u_j - 2u_{j-1} + u_{j-2}\right)$$

   applying the modified wave number analysis gives

$$ik^* - ik^* e^{-ik\Delta x} = \frac{1}{\Delta x}\left(1 - 2e^{-ik\Delta x} + e^{-2ik\Delta x}\right)$$

   which reduces to

$$ik^* = \frac{1 - 2cos(k\Delta x) + cos(2k\Delta x) + i(2sin(k\Delta x) - sin(2k\Delta x))}{\Delta x(1 - cos(k\Delta x) + isin(k\Delta x))}$$

3. Applying the representative equation $u'_n = \lambda u_n + ae^{\mu hn}$ to

$$u_{n+1} = 4u_n - 3u_{n-1} - 2h(u')_{n-1}$$

$$Eu_n - 4u_n + 3E^{-1}u_n + 2h\left(\lambda E^{-1}u_n\right) = -2hE^{-1}ae^{\mu hn}$$

giving us

(a)

$$P(E)u_n = Q(E)ae^{\mu hn}$$

with

$$\begin{aligned} P(E) &= E - 4 + (3 + 2h\lambda)E^{-1} \\ Q(E) &= -2hE^{-1} \end{aligned}$$

(b) The σ roots are obtained by letting $P(\sigma) = 0$

$$\begin{aligned} \sigma - 4 + (3 + 2h\lambda)\sigma^{-1} &= 0 \\ \sigma^2 - 4\sigma + (3 + 2h\lambda) &= 0 \end{aligned}$$

which gives us the two roots

$$\sigma_{1,2} = 2 \pm \sqrt{1 - 2h\lambda}$$

using $\sqrt{1 - 2h\lambda} = 1 - \frac{1}{2}(2h\lambda) - \frac{1}{8}(2h\lambda)^2 - \frac{1}{16}(2h\lambda)^3 + \cdots$

$$\begin{aligned} \sigma_1 &= 3 - h\lambda + \cdots \\ \sigma_2 &= 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{2}(h\lambda)^3 + \cdots \end{aligned}$$

(c) Letting $h\lambda = 0$ we see that $\sigma_1 = 3$ and $\sigma_2 = 1$ and therefore $\sigma_2 = 2 - \sqrt{1 - h\lambda}$ is the principal root and $\sigma_1 = 2 + \sqrt{1 - h\lambda}$ is the spurious root.

(d) Using the series expansion of

$$e^{\lambda h} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \cdots$$

we have $er_\lambda = -\frac{1}{3}(h\lambda)^3$ showing a 2nd order method.

4. Consider the method

$$\begin{aligned}
\tilde{u}_{n+\frac{1}{3}} &= u_n + \tfrac{h}{3}(u')_n \\
\widehat{u}_{n+\frac{1}{2}} &= u_n + \tfrac{h}{2}(\tilde{u}')_{n+\frac{1}{3}} \\
u_{n+1} &= u_n + h(\widehat{u}')_{n+\frac{1}{2}}
\end{aligned}$$

applied to the representative equation

$$u' = \lambda u$$

To identify the characteristic matrix $[P(E)]$ operator as discussed in class

(a)

$$\begin{bmatrix} 1 & 0 & -(1+\frac{h\lambda}{3}) \\ -\frac{h\lambda}{2} & 1 & -1 \\ 0 & -h\lambda & (E-1) \end{bmatrix} \begin{bmatrix} \tilde{u}_{n+\frac{1}{3}} \\ \widehat{u}_{n+\frac{1}{2}} \\ u_n \end{bmatrix} = 0$$

or if you ignored the hint

$$\begin{bmatrix} E^{\frac{1}{3}} & 0 & -(1+\frac{h\lambda}{3}) \\ -E^{\frac{1}{3}}\frac{h\lambda}{2} & E^{\frac{1}{2}} & -1 \\ 0 & -h\lambda E^{\frac{1}{2}} & (E-1) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \widehat{u} \\ u_n \end{bmatrix} = 0$$

in either case

$$[P(E)]\vec{u}_n = 0$$

(b) The characteristic polynomial $P(\sigma)$ = determinant of $[P(\sigma)]$ gives

$$P(\sigma) = \sigma - (1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3) = 0$$

Resulting in the σ root

$$\sigma = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3$$

(c) Then $er_\lambda = \frac{1}{24}(h\lambda)^4$, a 3rd Order accurate method.