Numerically Generated Chaos In Iterative Solution to Nonlinear ODEs

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This short note is not intended to be a complete treatment of nonlinear dynamics and chaos theory. The reader can find numerous texts on those subjects and will no doubt find more rigorous developments and explanations. The purpose for introducing these concepts here within the context of relaxation is to demonstrate that our classical way of thinking about relaxation applied to linear problems may lead us to conclusions which are not strictly valid for nonlinear problems. Typically, one uses results from linear analysis as guidelines for nonlinear problems. The stability and convergence characteristics of methods as analyzed by linear theory guides us in choices for nonlinear applications. But one can question whether linear analysis is strictly extendedable to nonlinear problems, where does the linear analysis breakdown, do linear and nonlinear problems behave consistently, along with a hosts of other such questions. Although we will not attempt here to answer such questions, these notes are intended to introduce the reader to an example where linear analysis doesn't tell the whole story. Where the behavior of the discrete approximation to nonlinear application can lead us astray.

Linear theory and numerical experimentation only provide a limited analysis of the effects of numerics and computational error on the computed solutions of a system of fluid dynamics equations. Typically, the numerical analyst relies on model problems, linear theory and past experience as a guideline in assessing and using numerical methods. Linear theory, while useful, can sometimes fail to uncover all the interesting characteristics of a method applied to a nonlinear problem. Recent work (Prüfer [1] and Lorenz [2]) has shown that simple numerical solution techniques for a nonlinear ODE can produce the equivalent of a chaotic "map" such as the logistic "map",

$$z^{n+1} = Rz^n(1 - z^n) \tag{1}$$

which for $3 \leq R \leq 4$ produces the asymptotic behavior of $z$ as shown in Fig. 1. A "map" is generated by iterating Eq. (1) from the initial value $z^0 = 0.25$ for 400 unplotted iterations and then ploting $z^i$ for $401 \leq i \leq 600$. For $R \leq 3.0$ the asymptotic behavior is a fixed point, i.e., $z^{i+1} = z^i$. At $R = 3.0$, a period-doubling cascade to chaos begins, followed by a region of mixed chaotic and periodic states up to $R = 4.0$ past which $z^i$ diverges to infinity, typically characterized as instability. In this context, period doubling refers to the bifurcation of a single fixed point, (i.e., one solution value $z^i$) into two solution values which are now two stable fixed points (i.e., $z^{i+1} \neq z^i$, but $z^{i+2} = z^i$). They are stable in the sense that, perturbations from those two fixed points always return uniquely to those two points for a given value of $R$ which lies at a doubling value. The next doubling leads to four fixed points and so on to values of $R$ where the solution $z^i$ lies in certain bounded regions, seen as the milky areas in Fig. 1. One can find many treatments and explanations of this map with discussions such as the significance of the "windows" (the fully black areas after the milky regions) and other interesting features. These are mostly irrelevant for our purposes here, we are only interested in the unusual nature of the map and as we shall see how it pertains to chosen iterative schemes for nonlinear ODE's.

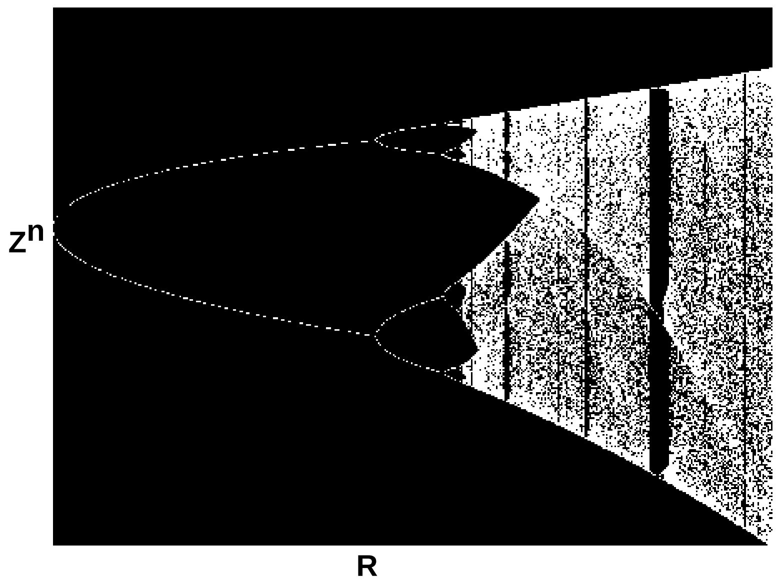


FIG. 1. Logistic Map $z^{n+1} = Rz^n(1 - z^n)$ for $R = 3.0 - 4.0$

To see how such behavior can arise in a numerical solution to an ODE consider the simple ODE

$$y_t = y(1 - y), \quad y(0) = y_0, \quad 0 \leq y_0 \leq 1 \tag{2}$$

which has the well behaved exact solution

$$y(t) = \frac{y_0}{y_0 + (1 - y_0)e^{-t}}. \tag{3}$$

A numerical solution may be obtained by applying Euler explicit time differencing,

$$y^{n+1} = y^n + \Delta t y^n \left(1 - y^n\right) = f(y^n). \tag{4}$$

Prufer [1] points out that Eq. (4) may be transformed into Eq. (1) by substituting $z^n = (\Delta t/(1 + \Delta t))y^n$ and $R = 1 + \Delta t$. Since we know Eq. (1) has the behavior

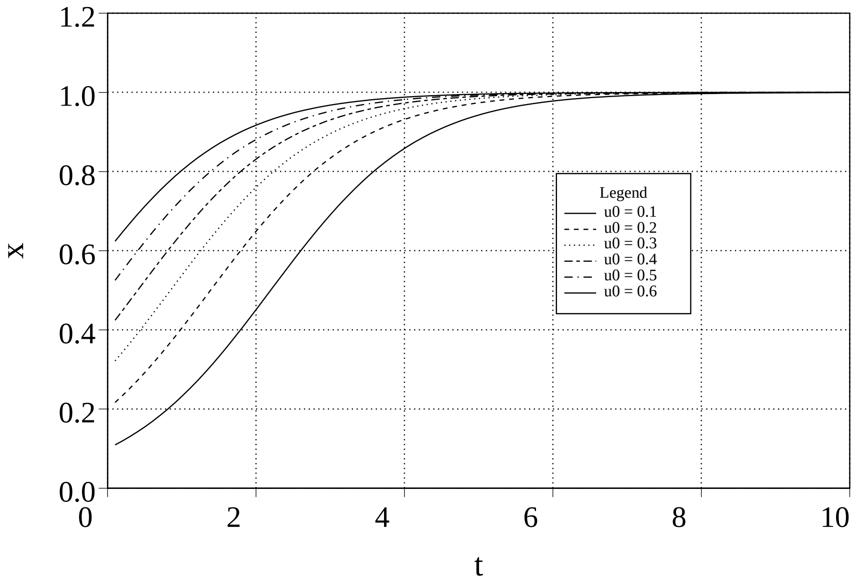


FIG. 2. Solutions From Different Initial Conditions for Eq. 2

shown in Fig. 1, then for some $\Delta t$, the solution given by Eq. (4) will behave chaotically for some initial conditions. The true solution, given by Eq. (3), is well-behaved, as is clear from Fig. 2, but the numerical behavior for some fixed $\Delta t$ will be as shown in Fig. 1.

Let's examine the linear stability of the discrete approximation given by Eq. (4). Linearizing Eq. (4) about the fixed point $y = 1$ yields

$$\widetilde{y}^{n+1} = (1 - \Delta t)\widetilde{y}^{n} \tag{5}$$

which has the linear stability bound $\Delta t \leq 2$. The location of the bifurcation to the first period two solution for the iterative map is determined by solving the nonlinear system

$$u = f(v) \quad v = f(u)$$

for $u, v$ in terms of $\Delta t$. The critical point $y_c$ is given by

$$y_c = \frac{(2 + \Delta t) \pm \sqrt{(\Delta t)^2 - 4}}{2\Delta t},$$

which only has real solutions for $\Delta t > 2.0$, coinciding with the linear stability bound. In this case the linear instability bound is also the point at which the discrete map approximation of the ODE bifurcates to a period two solution. Other discrete approximations to the Eq.(3) also have linear stability bounds coinciding with bifurcation boundaries, although this may not always be the case. For example, MacCormack's scheme has the linear stability bound $\Delta t \leq 2.0$ and bifurcation boundary $\Delta t > 2.0$, but has a different bifurcation map past the boundary than Euler explicit (i.e. the logistic map), see Fig. 3 for the lower half of the bifurcation diagram past $\Delta t = 2$.

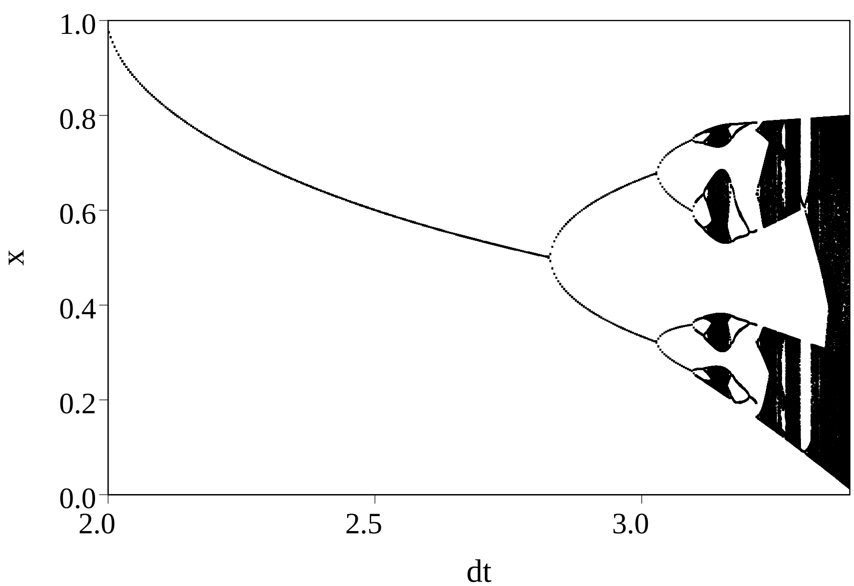


FIG. 3. Quadratic Like Chaotic Map For MacCormack's Scheme

Stuart [3] has studied systems of nonlinear PDE's and has shown that for a fairly general class of nonlinear reaction-diffusion equations, linearized instability leads to spurious periodic solutions in the nonlinear discretization.

These results are rather interesting in that they imply that there is a region of time steps where the solution is bounded but not convergent to the correct solution. Above a certain time step, one obtains the expected response of unbounded growth but between the linear stability bound and the unstable limit the numerical solution can behave as above, i.e. chaotically. This implies that in some situations, a new region of "unusual" stability can be defined. An example where one may take advantage of this behavior would be for stiff reaction equations where one part of a system lies in this bounded (but chaotic) region and other parts in the fixed point region. This opens up a new avenue of research into "nonlinear stability theory" which extends the usual "linear stability bounds". In practice, numerical analysts rely upon linear stability theory to determine limits for integrations. But this new area, especially when applied to basically nonlinear equations such as the Navier-Stokes equations, may lead to new understanding and algorithms.

1. M. Prüfer, *SIAM J.Appl. Math.* **45** (1985), 32.
2. E. Lorenz, *Physica* **35D** (1989), 299.
3. A. Stuart, *IMA J. of Num. Anal.* **9** (1989), 465.