AA214 – Solution Set 2 Answer Sheet
Answers on Taylor Tables and Truncatin Error

1. Problem 1: Answers available upon request.

2. Problem 2

   (a)

$$\alpha\Delta x(\delta_x u)_j - (-u_{j+2} + \beta u_{j+1} - \beta u_{j-1} + u_{j-2}) = \alpha\Delta x\ er_t$$

| | $u_j$ | $\Delta x\left(\frac{\partial u}{\partial x}\right)_j$ | $\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | $\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$ | $\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$ | $\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$ |
|---|---|---|---|---|---|---|
| $\alpha\Delta x(\delta_x u)_j$ | | $\alpha$ | | | | |
| $u_{j+2}$ | 1 | $\frac{(1)(2)^1}{1!}$ | $\frac{(1)(2)^2}{2!}$ | $\frac{(1)(2)^3}{3!}$ | $\frac{(1)(2)^4}{4!}$ | $\frac{(1)(2)^5}{5!}$ |
| $-\beta u_{j+1}$ | $-\beta$ | $\frac{(-\beta)(1)^1}{1!}$ | $\frac{(-\beta)(1)^2}{2!}$ | $\frac{(-\beta)(1)^3}{3!}$ | $\frac{(-\beta)(1)^4}{4!}$ | $\frac{(-\beta)(1)^5}{5!}$ |
| $\beta u_{j-1}$ | $\beta$ | $\frac{(\beta)(-1)^1}{1!}$ | $\frac{(\beta)(-1)^2}{2!}$ | $\frac{(\beta)(-1)^3}{3!}$ | $\frac{(\beta)(-1)^4}{4!}$ | $\frac{(\beta)(-1)^5}{5!}$ |
| $-u_{j-2}$ | $-1$ | $\frac{(-1)(-2)^1}{1!}$ | $\frac{(-1)(-2)^2}{2!}$ | $\frac{(-1)(-2)^3}{3!}$ | $\frac{(-1)(-2)^4}{4!}$ | $\frac{(-1)(-2)^5}{5!}$ |
| $\alpha\Delta x er_t$ | 0 | 0 | 0 | 0 | 0 | ? |

To get $4^{th}$ irder accuracy, the first 5 columns must sum to 0. The $1^{st}, 3^{rd}$, and $5^{th}$ columns are indentically 0, while setting the $2^{nd}$ and $4^{th}$ column to 0 give the two equations for the two unknowns $\alpha, \beta$

$$\alpha + 4 - 2\beta = 0$$
$$16 - 2\beta = 0$$

resulting in $\beta = 8$ and $\alpha = 12$ which gives the fourth order method with

$$er_t = \frac{1}{30}\Delta x^4\left(\frac{\partial^5 u}{\partial x^5}\right)_j$$

If $\beta = 4$, then $\alpha = 4$ and the fourth column does not sum to zero and the method is second order accurate.

3. From the Taylor table

| | $u_j$ | $\Delta x\cdot\left(\frac{\partial u}{\partial x}\right)_j$ | $\Delta x^2\cdot\left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | $\Delta x^3\cdot\left(\frac{\partial^3 u}{\partial x^3}\right)_j$ | $\Delta x^4\cdot\left(\frac{\partial^4 u}{\partial x^4}\right)_j$ |
|---|---|---|---|---|---|
| $a\cdot\Delta x\cdot\left(\frac{\partial u}{\partial x}\right)_{j-1}$ | | $a$ | $a\cdot(-1)\cdot\frac{1}{1}$ | $a\cdot(-1)^2\cdot\frac{1}{2}$ | $a\cdot(-1)^3\cdot\frac{1}{6}$ |
| $\Delta x\cdot\left(\frac{\partial u}{\partial x}\right)_j$ | | 1 | | | |
| $-d\cdot u_{j-1}$ | $-d$ | $-d\cdot(-1)\cdot\frac{1}{1}$ | $-d\cdot(-1)^2\cdot\frac{1}{2}$ | $-d\cdot(-1)^3\cdot\frac{1}{6}$ | $-d\cdot(-1)^4\cdot\frac{1}{24}$ |
| $-c\cdot u_j$ | $-c$ | | | | |
| $-b\cdot u_{j+1}$ | $-b$ | $-b\cdot(1)\cdot\frac{1}{1}$ | $-b\cdot(1)^2\cdot\frac{1}{2}$ | $-b\cdot(1)^3\cdot\frac{1}{6}$ | $-b\cdot(1)^4\cdot\frac{1}{24}$ |

the following equation has been constructed to maximize the order of accuracy

(a)

$$\begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ -2 & -1 & 0 & -1 \\ 3 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

This has the solution $[a, b, c, d] = [2, 1, 4, -5]/4$, so the scheme can be expressed as

$$2(\delta_x u)_{j-1} + 4(\delta_x u)_j - \frac{1}{\Delta x}[-5u_{j-1} + 4u_j + u_{j+1}] = O(\Delta x^3)$$

The Taylor series error of this difference scheme is

$$\begin{aligned} er_t &= \left(2 \cdot (-1)^3 \cdot \frac{1}{6} + 5 \cdot (-1)^4 \cdot \frac{1}{24} - 1 \cdot (1)^4 \cdot \frac{1}{24}\right) \frac{\Delta x^3}{4} \left(\frac{\partial^4 u}{\partial x^4}\right)_j \\ &= -\frac{\Delta x^3}{24} \left(\frac{\partial^4 u}{\partial x^4}\right)_j \end{aligned}$$

Answers on Modified Wave Numbers

4. Find the expression for the modified wave number in the following centered difference approximations to $(\delta_x u)_j$ in terms of $\Delta x$ and $k$. This is done just as done in class where we let $u_j = e^{ikj\Delta x}$. (Cast the results in terms of $sin(k\Delta x)$ and $cos(k\Delta x)$).

(a) $(\delta_x u)_j = (u_{j+1} - u_{j-1})/(2\Delta x)$
We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(e^{+ik\Delta x} - e^{-ik\Delta x}\right)/(2\Delta x)$$

which give us

$$ik' = i\frac{sin(k\Delta x)}{\Delta x}$$

(b) $(\delta_x u)_j = (-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2})/(12\Delta x)$
We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(-e^{2ik\Delta x} + 8e^{+ik\Delta x} - 8e^{-ik\Delta x} + e^{-2ik\Delta x}\right)/(12\Delta x)$$

which give us

$$ik' = i\frac{\left(\frac{4}{3}sin(k\Delta x) - \frac{1}{6}sin(2k\Delta x)\right)}{\Delta x}$$

(c) $\frac{1}{6}\left((\delta_x u)_{j+1} + 4(\delta_x u)_j + (\delta_x u)_{j-1}\right) = (u_{j+1} - u_{j-1})/(2\Delta x)$
We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\frac{1}{6}e^{ikj\Delta x}\left(ik' e^{+ik\Delta x} + 4ik' + ik' e^{-ik\Delta x}\right) = e^{ikj\Delta x}\left(e^{+ik\Delta x} - e^{-ik\Delta x}\right)/(2\Delta x)$$

which give us

$$\frac{1}{3}\left(2 + cos(k\Delta x)\right) ik' = i\frac{sin(k\Delta x)}{\Delta x}$$

or

$$ik' = i\frac{3sin(k\Delta x)}{\Delta x(2 + cos(k\Delta x))}$$

5. Find the expression for the modified wave number in the following one sided difference approximations to $(\delta_x u)_j$ in terms of $\Delta x$ and $k$. In this case there will be real and imaginary parts to the modified wave number. (Cast the results in terms of $sin(k\Delta x)$ and $cos(k\Delta x)$).

(a) $(\delta_x u)_j = (u_j - u_{j-1})/\Delta x$ We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(1 - e^{-ik\Delta x}\right)/(\Delta x)$$

which give us

$$ik' = \frac{1 - cos(k\Delta x)}{\Delta x} + i\frac{sin(k\Delta x)}{\Delta x}$$

(b) $(\delta_x u)_j = (3u_j - 4u_{j-1} + u_{j-2})/(2\Delta x)$
We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(3 - 4e^{-ik\Delta x} + e^{-2ik\Delta x}\right)/(2\Delta x)$$

which give us

$$ik' = \frac{3 - 4cos(k\Delta x) + cos(2k\Delta x)}{2\Delta x} + i\frac{4sin(k\Delta x) - sin(2k\Delta x)}{2\Delta x}$$

(c) $2(\delta_x u)_j + (\delta_x u)_{j-1} = (u_{j+1} + 4u_j - 5u_{j-1})/(2\Delta x)$
We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$e^{ikj\Delta x}\left(2ik' + ik' e^{-ik\Delta x}\right) = e^{ikj\Delta x}\left(e^{+ik\Delta x} + 4 - 5e^{-ik\Delta x}\right)/(2\Delta x)$$

which give us

$$(2 + cos(k\Delta x) - isin(k\Delta x))\, ik' = \frac{2 - 2cos(k\Delta x) + 3isin(k\Delta x)}{\Delta x}$$

or

$$ik' = \frac{2 - 2cos(k\Delta x) + 3isin(k\Delta x)}{(2 + cos(k\Delta x) - isin(k\Delta x))\,\Delta x}$$

You can avoid doing all the complex algebra and just evaluate these terms in complex arithmetic for the next problem.